Explicit second-order accurate method for the passive guaranteed simulation of port-Hamiltonian systems

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Abstract: This paper presents a method for the passive guaranteed simulation of a class of finite-dimensional nonlinear port-Hamiltonian systems. This method combines two processes to reach both the second order accuracy and explicit computations. First, we design a one-step two-stage implicit numerical method for Port-Hamiltonian systems that preserves passivity. Second, a change of state is proposed to yield an explicit computation. It requires assumptions on the Hamiltonian variations. The complete method is illustrated on two basic examples for which these assumptions are fulfilled.

Keywords: energy storage, port-Hamiltonian systems, nonlinear model, partial differential equations, passive simulation, consistency.

1. INTRODUCTION

Passive simulation is an important area of research as it guarantees the numerical stability for nonlinear systems.

Several methods are available to compute passive simulations, including for the sensitive issue of Hamiltonian mechanical systems which are conservative. They include Digital Waveguides, Wave Digital Filters and energy-conserving finite difference schemes (see Vilain (2010); Bilbao (2009); Julius (2010) for a review). In this work, we consider the port-Hamiltonian formulation which provides passive descriptions (decomposed into conservative and dissipative parts) of physical systems and of their combinations, in the continuous time domain Duindam (2009); Van der Schaft (2014).

Concerning Hamiltonian systems, numerous works have been done to compute passive simulation including for high-order accuracy Munthe-Kaas (1998); Del Buono (2002); Iserles (2000). For port-Hamiltonian systems, a numerical scheme based on a discrete definition of the Hamiltonian gradient has been described by Yalin (2015); Aoues (2014); Falaize (2014). In general, this method is not second-order accurate and is implicit. The computation of the implicit method needs Newton-Raphson type algorithm in which convergence issues may appear. Also, a high sampling frequency is needed to decrease the consistency error in the case of a one-order numerical scheme. Whatever, those limitations increase the computing time.

This paper presents a method for the passive guaranteed simulation of finite-dimensional nonlinear port-Hamiltonian systems. It combines two processes to reach both the second order accuracy and explicit computations: a one-step two-stage implicit numerical method for Port-Hamiltonian systems and a change of state to yield an explicit computation. This method is applied to a particular class of finite-dimensional port-Hamiltonian system and requires a strong assumption on the Hamiltonian variations for the implicit process.

In the first part of this paper, recalls on Port-Hamiltonian Systems and passive-guaranteed simulations are done including a study of the order of consistency. In a second part, we present the explicit second-order accurate method with two examples. Finally, results of simulation are described.

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2. RECALLS ON PORT-HAMILTONIAN SYSTEMS AND PASSIVE-GUARANTEED SIMULATIONS

2.1 Port-Hamiltonian systems

This section introduces a class of finite dimensional port-Hamiltonian System. A general presentation can be found in Duindam (2009).

A port-Hamiltonian system of state x, input u and output y can be represented by the following differential equations :

$$\dot{\mathbf{x}} = (\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x}))\partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \quad (1)$$

$$\mathbf{y} = \mathbf{G}(\mathbf{x})^T \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{D}(\mathbf{x})\mathbf{u} \quad (2)$$

where the positive definite function H denotes the energy of the system with respect to the state, where matrices $\mathbf{J}$ and $\mathbf{D}$ are skew-symmetric and where $\mathbf{R}$ is positive definite ($\mathbf{R} \geq 0$). This formulation guarantees the passivity as it naturally encodes the power balance. Indeed, as $\mathbf{J}$ and $\mathbf{D}$ are skew-symmetric, we have $\partial_{\mathbf{X}} H(\mathbf{x})^T \mathbf{J}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) = 0$, $\mathbf{u}^T \mathbf{D}(\mathbf{x})\mathbf{u} = 0$ and then,

$$\dot{H} = \underbrace{\partial_{\mathbf{X}} H^T \dot{\mathbf{x}}}_{\text{Energy variation}} = - \underbrace{\partial_{\mathbf{X}} H^T \mathbf{R} \partial_{\mathbf{X}} H}_{\substack{>0 \\ \text{Dissipated Power}}} + \underbrace{\mathbf{y}^T \mathbf{u}}_{\text{Entering Power}} \quad (3)$$

2.2 Discrete-gradient based numerical method

In this paper, we consider a zero-order hold for input $\mathbf{u}$ ($\mathbf{u}(t) = \mathbf{u}_n$ for $t \in [t_n, t_{n+1}[$) and a sampling period h.

In order to obtain the discrete-time formulation of a port-Hamiltonian conserving the property of passivity, we can use a discrete gradient based method already introduced in Yalin (2015); Aoues (2014); Falaize (2014).

The derivative of the state in eq. (1) is approximated by forward Euler as

$$\dot{\mathbf{x}} \cong \frac{\delta \mathbf{x}_n}{h} = \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{h}. \quad (4)$$

Also, to ensure the power balance eq. (3) in the discrete-time case, the discrete version of $\partial_{\mathbf{X}} H^T \dot{\mathbf{x}}$ must equal the energy variation between two steps:

$$\partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n)^T \delta \mathbf{x}_n = H(\mathbf{x}_n + \delta \mathbf{x}_n) - H(\mathbf{x}_n) \quad (5)$$

where $\partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n)$ denotes the discrete gradient of the Hamiltonian. For a N-dimensional system, the energy variation can be decomposed into a sum of N differences, as for example:

$$\begin{aligned} H(\mathbf{x} + \delta \mathbf{x}) - H(\mathbf{x}) = {} & H(\mathbf{x} + \delta \mathbf{x}) - H(x_1, x_2 + \delta x_2, ...) \\ & + H(x_1, x_2 + \delta x_2, ...) - H(x_1, x_2, x_3 + \delta x_3, ...) \\ & + \dots \\ & + H(x_1, ..., x_{N-1}, x_N + \delta x_N) - H(\mathbf{x}). \end{aligned} \quad (6)$$

This decomposition depends on the chosen increment order χ of δx and leads to a particular version of discrete Hamiltonian gradient that verifies eq. (5):

$$\lceil \partial^d_{\mathbf{X}} H(\mathbf{x}, \delta \mathbf{x}) \rceil_\chi = \begin{bmatrix} \frac{H(x_1 + \delta x_1, ...) - H(x_1, x_2 + \delta x_2, ...)}{\delta x_1} \\ \dots \\ \frac{H(x_1, ..., x_{N-1}, x_N + \delta x_N) - H(x_1, ..., x_{N-1}, x_N)}{\delta x_N} \end{bmatrix} \quad (7)$$

One can define a symmetrized version of the discrete gradient by averaging the solution for every path:

$$\partial^d_{\mathbf{X}} H(\mathbf{x}, \delta \mathbf{x}) = \frac{1}{N!} \sum_\chi \lceil \partial^d_{\mathbf{X}} H(\mathbf{x}, \delta \mathbf{x}) \rceil_\chi. \quad (8)$$

Finally, with (4) and (8) the discrete port-Hamiltonian system is described by:

$$\frac{\delta \mathbf{x}_n}{h} = (\mathbf{J}(\mathbf{x}_n) - \mathbf{R}(\mathbf{x}_n)) \partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n) + \mathbf{G}(\mathbf{x}_n) \mathbf{u}_n \quad (9)$$

$$\mathbf{y}_n = \mathbf{G}(\mathbf{x}_n)^T \partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n) + \mathbf{D}(\mathbf{x}_n) \mathbf{u}_n \quad (10)$$

where $\frac{\delta \mathbf{x}_n}{h} = \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{h} = \mathbf{s}_n$ and the slope $\mathbf{s}_n$ is this estimated state derivative. Thus, the discrete model leads to a one-step one-stage implicit and passive method for simulating:

$$\mathcal{M}_1 \begin{cases} \mathbf{s}_n &= (\mathbf{J}(\mathbf{x}_n) - \mathbf{R}(\mathbf{x}_n)) \partial^d_{\mathbf{X}} H(\mathbf{x}_n, h\mathbf{s}_n) + \mathbf{G}(\mathbf{x}_n) \mathbf{u}_n \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + h \mathbf{s}_n \end{cases}$$

2.3 Numerical analysis

This section is devoted to study the consistency order of the above numerical method $\mathcal{M}_1$. In the following, we denote $\mathbf{S} = \mathbf{J} - \mathbf{R}$ and the subscript n is omitted to simplify the notations . The notation $f(h) = \circ(g(h))$ is equivalent to $\lim_{h \to 0} \frac{f(h)}{g(h)} = 0$.

As proved in Demailly (2006) for the general case, we can assert that the order of the method is at least ℓ if, and only if:

$$\partial^i_h \mathbf{s}\big|_{h=0} = \frac{1}{1+i} \partial^i_t (\mathbf{S}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \mathbf{u})$$

for $i \leq \ell - 1$.

Theorem 1. The numerical method $\mathcal{M}_1$ is consistent. (Order 1)

Proof. Considering that $H(\mathbf{x})$ is infinitely differentiable, we can write the Taylor series of the multi-variable Hamiltonian function:

$$H(\mathbf{x} + \delta \mathbf{x}) = H(\mathbf{x}) + \partial_{\mathbf{X}} H(\mathbf{x})^T \delta \mathbf{x} + \frac{1}{2} \delta \mathbf{x}^T \partial^2_{\mathbf{X}} H(\mathbf{x})^T \delta \mathbf{x} + \circ(|\delta \mathbf{x}|^2) \quad (11)$$

where $\partial^2_{\mathbf{X}} H$ is the Hessian matrix of the Hamiltonian. With (5) and (11), we have,

$$\partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}) = \partial_{\mathbf{X}} H(\mathbf{x}_n) + \frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}_n) \delta \mathbf{x} + \circ(\delta \mathbf{x}). \quad (12)$$

It comes directly from ($\mathcal{M}_1$) and (12) that,

$$\mathbf{s}\big|_{h=0} = \mathbf{S}(\mathbf{x}) \partial^d_{\mathbf{X}} H(\mathbf{x}, 0) + \mathbf{G}(\mathbf{x}) \mathbf{u} \quad (13)$$

$$= \mathbf{S}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \mathbf{u} \quad (14)$$

This proves that the method is consistent.

Theorem 2. The numerical method $\mathcal{M}_1$ is second-order accurate if and only if $\mathbf{S} = \mathbf{J} - \mathbf{R}$ and $\mathbf{G}$ are independent of $\mathbf{x}$.

Proof. The order 1 proves that $\mathbf{s} = \dot{\mathbf{x}} + \circ(\mathbf{1})$. The equation (12) can be rewritten:

$$\partial^d_{\mathbf{X}} H(\mathbf{x}, \mathbf{s}h) = \partial_{\mathbf{X}} H(\mathbf{x}) + \frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}) \mathbf{s} h + \circ(\mathbf{h})$$

Thus,

$$\partial_h \mathbf{s} = \mathbf{S}(\mathbf{x}) \partial_h (\partial_{\mathbf{X}} H(\mathbf{x}) + \frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}) \mathbf{s} h + \circ(\mathbf{h})) \quad (15)$$

$$= \mathbf{S}(\mathbf{x}) (\frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}) \mathbf{s} + \circ(\mathbf{1})) \quad (16)$$

and,

$$\partial_h \mathbf{s}\big|_{h=0} = \frac{1}{2} \mathbf{S}(\mathbf{x}) \partial^2_{\mathbf{X}} H(\mathbf{x}) \dot{\mathbf{x}} \quad (17)$$

$$= \frac{1}{2} \mathbf{S}(\mathbf{x}) \partial_t (\partial_{\mathbf{X}} H(\mathbf{x})). \quad (18)$$

Finally,

$$\partial_h \mathbf{s}\big|_{h=0} = \frac{1}{2} \partial_t (\mathbf{S}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \mathbf{u})$$

if and only if $\mathbf{S} = \mathbf{J} - \mathbf{R}$ and $\mathbf{G}$ are independent of $\mathbf{x}$.

3. EXPLICIT SECOND-ORDER ACCURATE METHOD

This section presents an explicit second-order accurate method combining two processes:

(1) a one-step two-stage numerical method,
(2) a state change.

3.1 Second-order accurate method

We introduce the following one-step two-stage method for $\alpha \in [0;1[$.

$$\mathcal{M}_{2I}\begin{cases} \mathbf{s}_1 = \mathbf{S}(\mathbf{x}_n)\partial^d_{\mathbf{X}}H(\mathbf{x}_n,\mathbf{s}_1\alpha h) + \mathbf{G}(\mathbf{x}_n)\mathbf{u}_n \\ \mathbf{y}_1 = \mathbf{G}(\mathbf{x}_n)^T\partial^d_{\mathbf{X}}H(\mathbf{x}_n,\mathbf{s}_1\alpha h) + \mathbf{D}(\mathbf{x}_n)\mathbf{u}_n \\ \\ \mathbf{x}_\alpha = \mathbf{x}_n + \mathbf{s}_1\alpha h \\ \mathbf{x}_\beta = \mathbf{x}_n + \mathbf{s}_1\beta h \\ \\ \mathbf{s}_2 = \mathbf{S}(\mathbf{x}_\beta)\partial^d_{\mathbf{X}}H(\mathbf{x}_\alpha,\mathbf{s}_2(1-\alpha)h) + \mathbf{G}(\mathbf{x}_\beta)\mathbf{u}_n \\ \mathbf{y}_2 = \mathbf{G}(\mathbf{x}_\beta)^T\partial^d_{\mathbf{X}}H(\mathbf{x}_\alpha,\mathbf{s}_2(1-\alpha)h) + \mathbf{D}(\mathbf{x}_\beta)\mathbf{u}_n \\ \\ \mathbf{x}_{n+1} = \mathbf{x}_n + \alpha h\mathbf{s}_1 + (1-\alpha)h\mathbf{s}_2 \\ \mathbf{y}_n = \alpha\mathbf{y}_1 + (1-\alpha)\mathbf{y}_2 \end{cases}$$

This method is derived from a Runge-Kutta algorithm for the $\mathbf{x}_\beta$ point. On one hand, the $\mathbf{x}_\alpha$ point is used to ensure the power balance through the discrete gradient. On the other hand, the $\mathbf{x}_\beta$ is used to cancel out the second order error.

Theorem 3. The numerical method $\mathcal{M}_{2I}$ is second-order accurate if and only if $\beta = \frac{1}{2(1-\alpha)}$.

Proof. As for $(\mathcal{M}_1)$, this method is consistent (see proof in (2.3)) . We have

$$\partial_h\mathbf{s}_1 = \mathbf{S}(\mathbf{x})\partial_h(\partial_{\mathbf{X}}H(\mathbf{x}) + \frac{1}{2}\partial^2_{\mathbf{X}}H(\mathbf{x})\mathbf{s}_1\alpha h + \circ(\mathbf{h})) \tag{19}$$

$$= \frac{1}{2}\alpha\mathbf{S}(\mathbf{x})\partial^2_{\mathbf{X}}H(\mathbf{x})\mathbf{s}_1 + \circ(\mathbf{1}) \tag{20}$$

$$\tag{21}$$

and

$$\begin{aligned}\partial_h\mathbf{s}_2 &= \beta s_1\partial_{\mathbf{X}}\mathbf{S}(\mathbf{x}_\beta)\partial^d_{\mathbf{X}}H(\mathbf{x}_\alpha,\mathbf{s}_2(1-\alpha)h) \\ &+ \mathbf{S}(\mathbf{x}_\beta)\partial_h(\partial_{\mathbf{X}}H(\mathbf{x}_\alpha) + \frac{1}{2}\partial^2_{\mathbf{X}}H(\mathbf{x}_\alpha)\mathbf{s}_2(1-\alpha)h + \circ(\mathbf{h})) \\ &+ \beta\mathbf{s}_1\partial_{\mathbf{X}}\mathbf{G}(\mathbf{x}_\beta)\mathbf{u}_n \qquad (22)\\ &= \beta\mathbf{s}_1\partial_{\mathbf{X}}\mathbf{S}(\mathbf{x}_\beta)\partial_{\mathbf{X}}H(\mathbf{x}_\alpha) \\ &+ \mathbf{S}(\mathbf{x}_\beta)(\alpha\mathbf{s}_1\partial^2_{\mathbf{X}}H(\mathbf{x}_\alpha) + \frac{1}{2}\partial^2_{\mathbf{X}}H(\mathbf{x}_\alpha)\mathbf{s}_2(1-\alpha) + \circ(\mathbf{1})) \\ &+ \beta\mathbf{s}_1\partial_{\mathbf{X}}\mathbf{G}(\mathbf{x}_\beta)\mathbf{u}_n. \qquad (23)\end{aligned}$$

Finally,

$$\begin{aligned}\partial_h\mathbf{s}|_{h=0} &= \alpha\,\partial_h\mathbf{s}_1|_{h=0} + (1-\alpha)\,\partial_h\mathbf{s}_2|_{h=0} \qquad (24)\\ &= (\frac{1}{2}\alpha^2 + (1-\alpha)(\alpha + \frac{1}{2}(1-\alpha)))\mathbf{S}(\mathbf{x})(\partial_{\mathbf{X}}H(\mathbf{x}))' \\ &+ (1-\alpha)\beta(\mathbf{S}(\mathbf{x}))'\partial_{\mathbf{X}}H(\mathbf{x}) + (1-\alpha)\beta(G(\mathbf{x}))'\mathbf{u} \qquad (25)\\ &= \frac{1}{2}\mathbf{S}(\mathbf{x})\partial_h(\partial_{\mathbf{X}}H(\mathbf{x})) \\ &+ (1-\alpha)\beta(\mathbf{S}(\mathbf{x}))'\partial_{\mathbf{X}}H(\mathbf{x}) + (1-\alpha)\beta(G(\mathbf{x}))'\mathbf{u} \qquad (26)\end{aligned}$$

This proves that $\partial_h\mathbf{s}|_{h=0} = \frac{1}{2}\partial_h(\mathbf{S}(\mathbf{x})\partial_{\mathbf{X}}H(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u})$ and then that the method is second-order accurate if and only if $\beta = \frac{1}{2(1-\alpha)}$.

Theorem 4. The numerical method $\mathcal{M}_{2I}$ guarantees the passivity.

Proof. We can write the following decomposition:

$$\begin{aligned}H(\mathbf{x}_{n+1}) - H(\mathbf{x}_n) &= H(\mathbf{x}_n + \alpha\mathbf{s}_1 + (\alpha-1)\mathbf{s}_2) - H(\mathbf{x}_n + \alpha\mathbf{s}_1) \\ &+ H(\mathbf{x}_n + \alpha\mathbf{s}_1) - H(\mathbf{x}_n) \qquad (27)\end{aligned}$$

From (5), it comes

$$H(\mathbf{x}_{n+1}) - H(\mathbf{x}_n) = -\alpha\mathcal{P}_1 - (1-\alpha)\mathcal{P}_2 + \mathbf{y}_n^T\mathbf{u} \tag{28}$$

where $\mathcal{P}_1 = \partial^d_{\mathbf{X}}H(\mathbf{x}_n,\mathbf{s}_1\alpha h)^T\mathbf{R}(\mathbf{x}_n)\partial^d_{\mathbf{X}}H(\mathbf{x}_n,\mathbf{s}_1\alpha h) \geq 0$ and $\mathcal{P}_2 = \partial^d_{\mathbf{X}}H(\mathbf{x}_\alpha,\mathbf{s}_2(1-\alpha)h)^T\mathbf{R}(\mathbf{x}_\beta)\partial^d_{\mathbf{X}}H(\mathbf{x}_\alpha,\mathbf{s}_2(1-\alpha)h) \geq 0$ are the dissipated powers between the points $\mathbf{x}_n$ and $\mathbf{x}_\alpha$, and the points $\mathbf{x}_\alpha$ and $\mathbf{x}_{n+1}$ respectively.

As for the first method $\mathcal{M}_1$, this method stays implicit for the computation of the slopes $\mathbf{s}_1$ and $\mathbf{s}_2$. However, choosing $\alpha = 0$ leads to an explicit formulation for the first slope $\mathbf{s}_1$.

3.2 Explicit method

In this section, we consider the case of a system with an Hamiltonian H that verifies the hypothesis:

$$sign(x_i)\partial_{x_i}H(\mathbf{x}) > 0 \text{ for } \mathbf{x} \in \mathcal{E} \tag{29}$$

where $\mathcal{E}$ is the configuration space of the system. With this hypothesis, we can define the following change of state:

$$\underline{\mathbf{x}}_\chi = \begin{pmatrix} sign(x_1)\sqrt{2}\sqrt{H(x_1,\dots,x_N) - H(0,x_2,\dots,x_N)} \\ \vdots \\ sign(x_N)\sqrt{2}\sqrt{H(0,\dots,0,x_N) - H(0,\dots,0)} \end{pmatrix}. \tag{30}$$

Analogously to the discrete Hamiltonian method 2.2, χ denotes a chosen order to set the state to zero . In this way, we can define a symmetrized version for the change of state:

$$\underline{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \frac{1}{N!}\sum_\chi \underline{\mathbf{x}}_\chi(\mathbf{x}) \tag{31}$$

Theorem 5. The port-Hamiltonian system with respect to the new state $\underline{\mathbf{x}}$ is described by:

$$\begin{aligned}\dot{\underline{\mathbf{x}}} &= (\underline{\mathbf{J}}(\underline{\mathbf{x}}) - \underline{\mathbf{R}}(\underline{\mathbf{x}}))\partial_{\underline{\mathbf{X}}}H(\underline{\mathbf{x}}) + \underline{\mathbf{G}}(\underline{\mathbf{x}})\mathbf{u} \\ \mathbf{y} &= \underline{\mathbf{G}}(\underline{\mathbf{x}})^T\partial_{\underline{\mathbf{X}}}H(\underline{\mathbf{x}}) + \underline{\mathbf{D}}(\underline{\mathbf{x}})\mathbf{u}\end{aligned} \tag{32}$$

with $H(\underline{\mathbf{x}}) = \frac{1}{2}\underline{\mathbf{x}}^T\underline{\mathbf{x}}$ and where denoting $g = f^{-1}$

$$\begin{aligned}\underline{\mathbf{J}}(\underline{\mathbf{x}}) &= \partial_{\mathbf{X}}\mathbf{f}\circ\mathbf{g}^T\mathbf{J}(\mathbf{g}(\underline{\mathbf{x}}))\partial_{\underline{\mathbf{X}}}\mathbf{f}\circ\mathbf{g} \\ \underline{\mathbf{R}}(\underline{\mathbf{x}}) &= \partial_{\mathbf{X}}\mathbf{f}\circ\mathbf{g}^T\mathbf{R}(\mathbf{g}(\underline{\mathbf{x}}))\partial_{\underline{\mathbf{X}}}\mathbf{f}\circ\mathbf{g} \\ \underline{\mathbf{G}}(\underline{\mathbf{x}}) &= \partial_{\mathbf{X}}\mathbf{f}\circ\mathbf{g}^T\mathbf{G}(\mathbf{g}(\underline{\mathbf{x}})) \\ \underline{\mathbf{D}}(\underline{\mathbf{x}}) &= \mathbf{D}(\mathbf{g}(\underline{\mathbf{x}}))\end{aligned}$$

Proof. Given the change of state $\underline{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{g} = \mathbf{f}^{-1}$, we have

$$\dot{\mathbf{x}} = \partial_{\underline{\mathbf{X}}}\mathbf{g}^T\dot{\underline{\mathbf{x}}} \tag{33}$$

$$\partial_{\underline{\mathbf{X}}}H(\underline{\mathbf{x}}) = \partial_{\underline{\mathbf{X}}}\mathbf{g}\partial_{\mathbf{X}}H(\mathbf{x}) \tag{34}$$

where $\partial_{\underline{\mathbf{X}}}\mathbf{g}$ denotes the Jacobian of $\mathbf{g}$. From eq. (1) and eq. (32), we have

$$\underline{\mathbf{J}}(\underline{\mathbf{x}}) = \partial_{\underline{\mathbf{X}}}\mathbf{g}^{-T}\mathbf{J}(\mathbf{g}(\underline{\mathbf{x}}))\partial_{\underline{\mathbf{X}}}\mathbf{g}^{-1} \tag{35}$$

$$\underline{\mathbf{R}}(\underline{\mathbf{x}}) = \partial_{\underline{\mathbf{X}}}\mathbf{g}^{-T}\mathbf{R}(\mathbf{g}(\underline{\mathbf{x}}))\partial_{\underline{\mathbf{X}}}\mathbf{g}^{-1} \tag{36}$$

$$\underline{\mathbf{G}}(\underline{\mathbf{x}}) = \partial_{\underline{\mathbf{x}}}\mathbf{g}^{-T}\mathbf{G}(\mathbf{g}(\underline{\mathbf{x}})) \quad (37)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{x}}) = \mathbf{D}(\mathbf{g}(\underline{\mathbf{x}})) \quad (38)$$

$$(39)$$

Finally, as $\mathbf{g} = \mathbf{f}^{-1}$, we have $\partial_{\underline{\mathbf{x}}}\mathbf{g}^{-1} = \partial_{\mathbf{x}} f \circ \mathbf{g}$

Theorem 6. With the new state, the computation of the slopes in the methods $\mathcal{M}_1$ and $\mathcal{M}_{2I}$ is explicit.

Proof. As the Hamiltonian with respect to the state is quadratic and separable ($H(\underline{\mathbf{x}}) = \frac{1}{2}\underline{\mathbf{x}}^T\underline{\mathbf{x}}$), the discrete gradient (8) can be rewritten $\partial^d_{\underline{\mathbf{x}}} H(\underline{\mathbf{x}}, \delta\underline{\mathbf{x}}) = \underline{\mathbf{x}} + \frac{1}{2}\delta\underline{\mathbf{x}}$. Thus (9) becomes

$$\delta\underline{\mathbf{x}} = h(\mathbf{I} - \frac{1}{2}h\underline{\mathbf{S}}(\underline{\mathbf{x}}))^{-1}(\underline{\mathbf{S}}(\underline{\mathbf{x}})\underline{\mathbf{x}} + \underline{\mathbf{G}}(\underline{\mathbf{x}})\mathbf{u})$$

or for the slope

$$\underline{\mathbf{s}} = (\mathbf{I} - \frac{1}{2}h\underline{\mathbf{S}}(\underline{\mathbf{x}}))^{-1}(\underline{\mathbf{S}}(\underline{\mathbf{x}})\underline{\mathbf{x}} + \underline{\mathbf{G}}(\underline{\mathbf{x}})\mathbf{u})$$

This formulation is explicit.

Finally, we can introduce a third method combining the change of state 30 and the one-step two-stage method $\mathcal{M}_{2I}$.

$$\mathcal{M}_{2E}\begin{cases} \mathbf{s}_1 = (\mathbf{I} - \frac{1}{2}\alpha h\underline{\mathbf{S}}(\underline{\mathbf{x}}_n))^{-1}(\underline{\mathbf{S}}(\underline{\mathbf{x}}_n)\underline{\mathbf{x}}_n + \underline{\mathbf{G}}(\underline{\mathbf{x}}_n)\mathbf{u}_n) \\ \mathbf{y}_1 = \quad \underline{\mathbf{G}}(\underline{\mathbf{x}}_n)^T\partial^d_{\underline{\mathbf{x}}} H(\underline{\mathbf{x}}_n, \mathbf{s}_1\alpha h) + \underline{\mathbf{D}}(\underline{\mathbf{x}}_n)\mathbf{u}_n \\ \\ \underline{\mathbf{x}}_\alpha = \underline{\mathbf{x}}_n + \mathbf{s}_1\alpha h \\ \underline{\mathbf{x}}_\beta = \underline{\mathbf{x}}_n + \mathbf{s}_1\beta h \\ \\ \mathbf{s}_2 = (\mathbf{I} - \frac{1}{2}(1-\alpha)h\underline{\mathbf{S}}(\underline{\mathbf{x}}_\beta))^{-1}(\underline{\mathbf{S}}(\underline{\mathbf{x}}_\beta)\underline{\mathbf{x}}_\alpha + \underline{\mathbf{G}}(\underline{\mathbf{x}}_\beta)\mathbf{u}) \\ \mathbf{y}_2 = \quad \underline{\mathbf{G}}(\underline{\mathbf{x}}_\beta)^T\partial^d_{\underline{\mathbf{x}}} H(\underline{\mathbf{x}}_\alpha, \mathbf{s}_2(1-\alpha)h) + \underline{\mathbf{D}}(\underline{\mathbf{x}}_\beta)\mathbf{u}_n \\ \\ \underline{\mathbf{x}}_{n+1} = \underline{\mathbf{x}}_n + \alpha h\mathbf{s}_1 + (1-\alpha)h\mathbf{s}_2 \\ \mathbf{y}_n = \quad \alpha\mathbf{y}_1 + (1-\alpha)\mathbf{y}_2 \end{cases}$$

4. EXAMPLES

4.1 Example 1: mechanical oscillator with a nonlinear dissipation

Consider the classical mass-spring-damper system in figure(1) where the energy is quadratic, $H(p,x) = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$ and the damping is nonlinear $F_{\texttt{damp}} = c\dot{x}^3$.

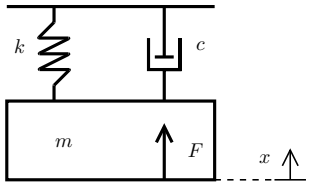

Fig. 1. Classical mass-spring-damper system under a force F.

The system is governed by the equation

$$\dot{p} = -kx - c\dot{x}^3 + F \quad (40)$$

It can be set under the form (1) with $\mathbf{x} = [p,x]^T$, $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{R}(\mathbf{x}) = \begin{pmatrix} \frac{c}{m^2}p^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{G} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The change of state (30) gives

$$\underline{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} p/\sqrt{m} \\ \sqrt{k}x \end{pmatrix}.$$

Thus, the system can be rewritten under the form (32) with

with $\underline{\mathbf{J}} = \begin{pmatrix} 0 & -\sqrt{(k/m)} \\ \sqrt{(k/m)} & 0 \end{pmatrix}$, $\underline{\mathbf{R}}(\mathbf{x}) = \begin{pmatrix} \frac{c}{m\sqrt{m}}\underline{p}^2 & 0 \\ 0 & 0 \end{pmatrix}$, $\underline{\mathbf{G}} = \begin{pmatrix} 1/\sqrt{m} \\ 0 \end{pmatrix}$ and $H(\underline{p},\underline{x}) = \frac{1}{2}\underline{p}^2 + \frac{1}{2}\underline{x}^2$.

4.2 Example 2: mechanical oscillator with a hardening spring

Consider the same system that in example (4.1) illustrated in figure (1) with a linear damping $F_{\texttt{damp}} = c\dot{x}$ and a nonlinear hardening spring $F_{\texttt{spring}} = sinh(x)$.

The system is governed by the equation

$$p = -sinh(x) - c\dot{x} + F$$

It can be set under the form (1) with $\mathbf{x} = [p,x]^T$, $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{G} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $H(p,x) = \frac{1}{2}p^2 + (cosh(x) - 1)$. The change of state (30) gives

$$\underline{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \dot{p} \\ sign(x)\sqrt{2}\sqrt{cosh(x) - 1} \end{pmatrix}$$

and

$$\partial_{\mathbf{x}}\mathbf{f} \circ \mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\underline{x}^2}{4} + 1} \end{pmatrix}.$$

with

$$\mathbf{g}(\underline{\mathbf{x}}) = sign(\underline{x})arccosh(\underline{x}^2/2 + 1)$$

Thus, the system can be rewritten under the form (32) with $\underline{\mathbf{J}}(\underline{\mathbf{x}}) = \begin{pmatrix} 0 & -\sqrt{\frac{\underline{x}^2}{4} + 1} \\ \sqrt{\frac{\underline{x}^2}{4} + 1} & 0 \end{pmatrix}$, $\underline{\mathbf{R}}(\mathbf{x}) = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$, $\underline{\mathbf{G}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $H(\underline{p},\underline{x}) = \frac{1}{2}\underline{p}^2 + \frac{1}{2}\underline{x}^2$.

4.3 Simulation and results:

This section presents results for two configurations of the system presented in the example (4.2):

(i) a conservative case ($\mathbf{u} = 0$ and $c = 0$),
(ii) a general case (with entering power and dissipation).

In both cases, the states at $t = 0$ are $p = 100\ kgms^{-1}$ and $x = sign(\underline{x})arccosh(\underline{x}^2/2 + 1) = 0\ m$.

For the conservative case (i), we use the $\mathcal{M}_{2E}$ method with a sampling rate $f_s = 1khz$. The position x is displayed is figure (2)(Top) using the change of variable $x = sign(\underline{x})arccosh(\underline{x}^2/2 + 1)$. In this configuration, the energy of the system must remain constant at $E_0 = H(\mathbf{x}(t = 0))$. The figure 2(Bottom) shows the error on the energy $H(\mathbf{x}) - E_0$ during the experiment. The error is near to the hardware error and confirms the passivity property of the method.

For the general case (ii), the input is constant $F = 100N$ and the dissipation is $c = 1Nsm^{-1}$. In order to make

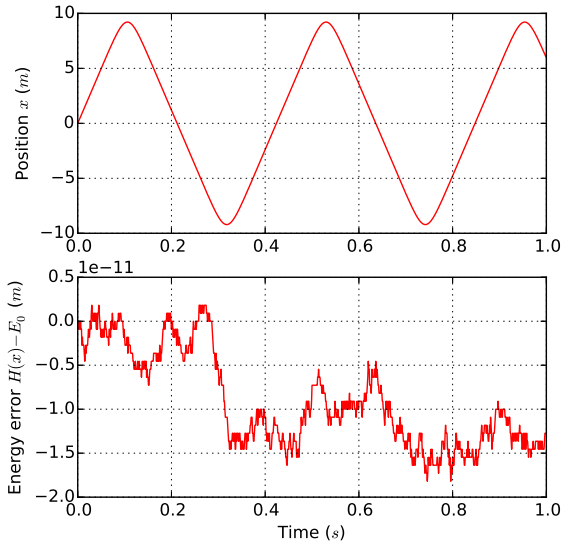

Fig. 2. Simulation of the system described in example (4.2) for a conservative configuration, with the explicit method $(\mathcal{M}_2)$ and $f_s = 1Khz$. (Top) Position of the mass x w.r.t. time. (Bottom) Error on the energy $H(\mathbf{x}) - E_0$.

comparisons, we simulate the system under the form (1) with a classical forward Euler method with $f_s = 1Mhz$ without any consideration on the passivity. With this high sampling frequency, we can considerate that this simulation is a good reference $(\mathcal{R})$ to evaluate the consistency error. The figure (3) shows the position for $(\mathcal{R})$ at $f_s = 1Mhz$ and $(\mathcal{M}_{2E})$ at $f_s = 1khz$.

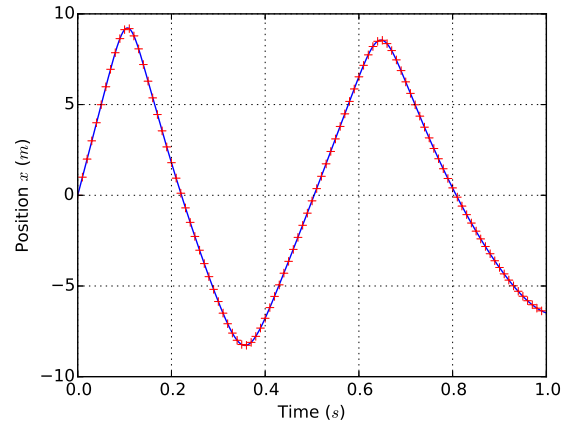

Fig. 3. Position of the mass x from a simulation of the system described in example (4.2) for a non-conservative configuration, with the explicit method $(\mathcal{M}_2)$ $f_s = 1Khz$ in red crosses and the reference at $f_s = 1Mhz$ in solid blue.

The figure (4) displays the averaged error between the reference $\mathcal{R}$ and the results from the methods:

(1) $(\mathcal{M}_1)$ (green solid line) with the explicit formulation,
(2) $(\mathcal{M}_{2E})$ (red dotted line),

with respect to the sampling frequency for the states p (top) and x (bottom). The logarithm scale highlights that the consistency order is order 1 for the first method and order 2 for the method $(\mathcal{M}_{2E})$. Results above $5khz$ are biased because of the reference $\mathcal{R}$ which is not the exact solution.

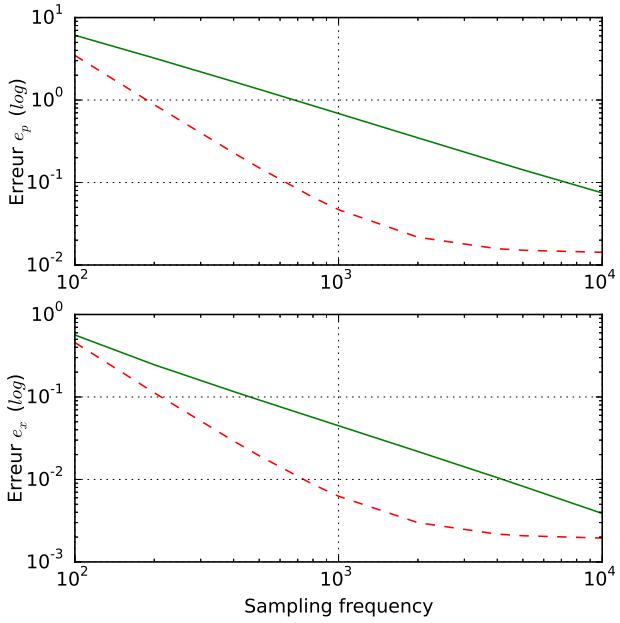

Fig. 4. Consistency error between the method $(\mathcal{M}_{2E})$ in red dotted line and the method $(\mathcal{M}_1)$ in green solid line. (Top) Consistency error (in logarithmic scale) for the state p.(Bottom) Consistency error (in logarithmic scale) for the state x.

5. CONCLUSION

In this work, we have presented an explicit second-order accurate method for the passive guaranteed simulation of port-Hamiltonian systems. First, we have described a multi-stage numerical method based on a discrete gradient. This method has been proved to be passive and second order accurate even if the system matrices depend on the state. Second, a change of state has been proposed to yield an explicit computation.

The explicit method is constrained by a strong hypothesis (29) on the Hamiltonian variation. In future works, we will study methods compatible with the case of periodic Hamiltonian (e.g. for the non-linear pendulum). Moreover, higher order methods conserving the passive and the explicit properties will be investigate.

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