A generic passive-guaranteed structure for elastoplatic friction models

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Context: Sound synthesis

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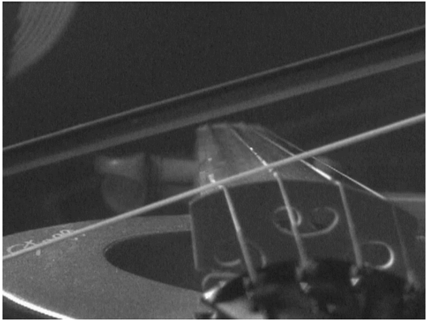
Bowed string dynamics: Nonlinear dynamical system with self-oscillations due to an alternance between stick and slip phases[1].



Source: https://www.youtube.com/watch?v=6JeyiM0YNo4 (ViolinB0W)

[1]Willemsen, S.,Bilbao, S., Serafin, S. (2019). Real-time implementation of an elasto-plastic friction model applied to stiff strings using finite-difference schemes. 22nd International Conference on Digital Audio Effects.

Problem Statement

Aims

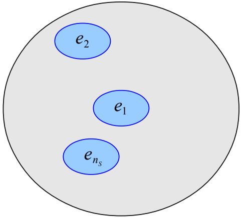
- Sound synthesis of a resonator with nonlinear interaction
- Ensure power balance and therefore computation stability

Framework: Port Hamiltonian Systems (PHS)

- PHS formulation
- Modularity: inteconnexion of several PHS is a PHS
- **Passive guaranteed simulation** (stability)

Outline

Port-Hamiltonian Systems



- Energy storing components: *(energy)*
  $E = \sum_{n=1}^{N} e_n \geq 0$

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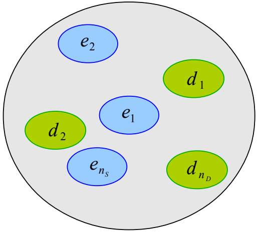

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- Dissipative components: *(dissipated power)*
  $P_D = \sum_{m=1}^{M} d_m \geq 0$

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Port-Hamiltonian Systems

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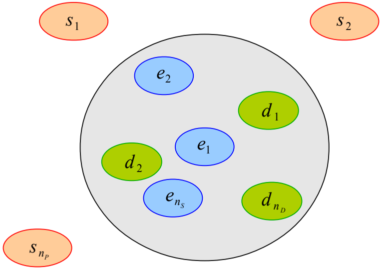

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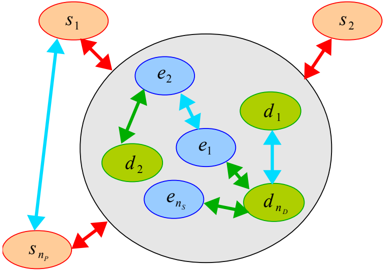
- External sources: *(external power)*
  $P_{\mathrm{ext}} = \sum_{p=1}^{P} s_p$

Port-Hamiltonian Systems



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- External sources: *(external power)*
  $P_{\mathrm{ext}} = \sum_{p=1}^{P} s_p$
- Conservatives connections *(power balance)*
  $\frac{dE}{dt} = -P_D + P_{\mathrm{ext}}$

Port-Hamiltonian Systems

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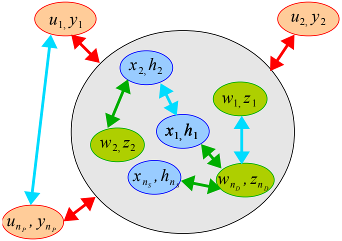
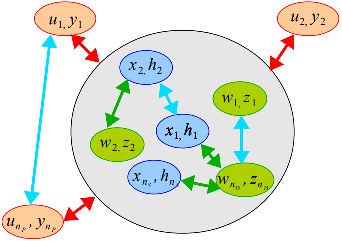


- Energy storing components: *(energy)*
  $E = H(x) = \sum_{n=1}^{N} H_n(x_n) \geq 0$
- Dissipative components: *(dissipated power)*
  $P_D = z(w)^T w = \sum_{m=1}^{M} z_m(w_m)\, w_m \geq 0$
  (effort × flow: force × velocity, tension × current, etc)
- External sources: *(external power)*
  $P_{\mathrm{ext}} = u^T y = \sum_{p=1}^{P} u_p y_p$
- Conservatives connections *(power balance)*
  $0 = \nabla H(x)^T \frac{\mathrm{d}x}{\mathrm{d}t} + z(w)^T \cdot w - u^T \cdot y$

Port-Hamiltonian Systems



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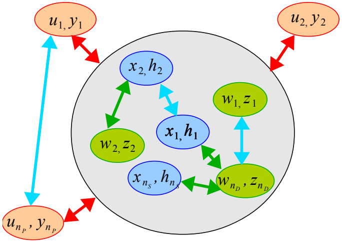
Port Hamiltonian Formulation

$$\begin{pmatrix} \frac{d\mathrm{x}}{dt} \\ \hline \mathrm{w} \\ \hline -\mathrm{y} \end{pmatrix} = S. \begin{pmatrix} \nabla H(\mathrm{x}) \\ \hline \mathrm{z}(\mathrm{w}) \\ \hline \mathrm{u} \end{pmatrix}$$

Power balance

Port-Hamiltonian Systems



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Port Hamiltonian Formulation

$$\underbrace{\begin{pmatrix} \frac{d\mathrm{x}}{dt} \\ \hline \mathrm{w} \\ \hline -\mathrm{y} \end{pmatrix}}_{B} = S.\underbrace{\begin{pmatrix} \nabla H(\mathrm{x}) \\ \hline \mathrm{z}(\mathrm{w}) \\ \hline \mathrm{u} \end{pmatrix}}_{A}$$

Power balance

$$\boxed{\begin{aligned} 0 &= A^T B \\ &= A^T S A \end{aligned}}$$

if $S = -S^T$

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Example: The string model

Linear string model[2]

$$\begin{pmatrix} \frac{\partial q}{\partial t}(\xi, t) \\ \frac{\partial v}{\partial t}(\xi, t) \\ \hline w_s(\xi, t) \\ \hline y_s(\xi, t) \end{pmatrix} = \left(\begin{array}{cc|c|c} 0 & \frac{1}{\mu}\frac{\partial}{\partial \xi} & 0 & 0 \\ \frac{1}{\mu}\frac{\partial}{\partial \xi} & 0 & \frac{-1}{\mu} & \frac{-1}{\mu} \\ \hline 0 & \frac{1}{\mu} & 0 & 0 \\ \hline 0 & \frac{1}{\mu} & 0 & 0 \end{array}\right) \begin{pmatrix} \frac{\partial \mathrm{H}_s}{\partial q}(q(\xi, t), v(\xi, t)) \\ \frac{\partial \mathrm{H}_s}{\partial v}(q(\xi, t), v(\xi, t)) \\ \hline z_s\left(w_s(\xi, t)\right) \\ \hline u_s(\xi, t) \end{pmatrix}$$

with

- For the storing components: $\frac{\partial \mathrm{H}_s}{\partial q} = T_0 q$ and $\frac{\partial \mathrm{H}_s}{\partial v} = \mu v$
- For the dissipating components: $w_s = v$ and $z_s\left(w_s(\xi, t)\right) = av$
- For the power source: $u_s(\xi, t) = -f(\xi, t)$ and $y_s(\xi, t) = v(\xi, t)$

[2] Villegas J.A., A port-Hamiltonian approach to distributed paremeter systems, PhD thesis, University of Twente, 2007.

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The bowed string

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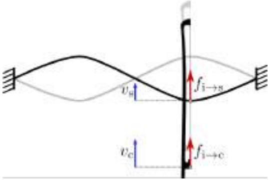

Friction model with two external ports

Control port with imposed velocity v_c and $f_{i\to c}$ the force exerted by the interaction.

Interaction port with string velocity v_s and $f_{i\to s}$ the force exerted on the string.

Power emitted thought the ports : $\mathrm{u}^T\mathrm{y} = f_{i\to s}v_s + f_{i\to c}v_c$.

Dupont friction model

Friction model

The interaction forces depend on the *relative* velocity $v_{\mathrm{rel}} = v_{\mathrm{c}} - v_{\mathrm{s}}$.

Elastoplastic friction model

The relative velocity is decomposed as $v_{\mathrm{rel}} = v_{\mathrm{e}} + v_{\mathrm{p}}$ with:

- Internal elastic state x_{e} associated with velocity $v_{\mathrm{e}} = \dot{x}_{\mathrm{e}}$ and potential energy $h_{\mathrm{e}}(x_{\mathrm{e}}) = \frac{k_{\mathrm{e}} x_{\mathrm{e}}^2}{2} \rightarrow$ storing component (reversible)
- Plastic behavior with associated velocity $v_{\mathrm{p}} \rightarrow$ dissipative component (irreversible)

Dupont friction model [3]

- Includes a compliance damping σ_c and a fluid damping σ_f.
- Is given as an implicit relation

$$\begin{cases} f_{i\to s} & = k_e x_e + \sigma_c \frac{\mathrm{d}x_e}{\mathrm{d}t} + \sigma_f v_{rel} \\ \frac{\mathrm{d}x_e}{\mathrm{d}t} & = v_{rel}\left(1 - \frac{|v_{rel}|\alpha(x_e, w_{rel})k_e}{v_{rel} f_{ss}(v_{rel})} x_e\right) \end{cases} \tag{1}$$

where the steady state friction force (Stribeck curve) is defined by:

$$f_{\mathrm{ss}}(v_{\mathrm{rel}}) = \frac{1}{k_{\mathrm{e}}}\left(f_C + (f_{\mathrm{S}} - f_C)\exp\left(-\left(\frac{v_{\mathrm{rel}}}{v_{\mathrm{S}}}\right)^2\right)\right), \tag{2}$$

and $\alpha(x_{\mathrm{e}}, v_{\mathrm{rel}}) \in (0, 1)$ is an adhesion map (shown next slide).

[3] Dupont, P., Hayward, V., Armstrong, B., Altpeter, F. (2002). Single state elastoplastic friction models. IEEE Transactions on automatic control, 47(5), 787-792.

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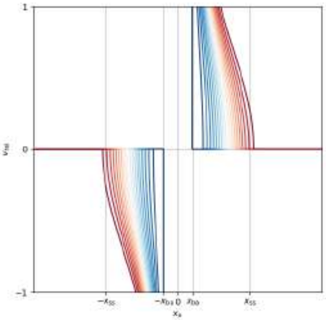
Adhesion map

Dissipation relative to the plastic behaviour:

$$\alpha(x_e, v_{rel}) = \begin{cases} 0 & \text{if} \quad \mathrm{sign}(x_e) \neq \mathrm{sign}(v_{rel}), \\ 0 & \text{if} \quad \mathrm{sign}(x_e) = \mathrm{sign}(v_{rel}), \quad |x_e| \leq x_{ba}, \\ \widehat{\alpha}(x_e, v_{rel}) & \text{if} \quad \mathrm{sign}(x_e) = \mathrm{sign}(v_{rel}), \quad x_{ba} < |x_e| < x_{ss}(v_{rel}), \\ 1 & \text{if} \quad \mathrm{sign}(x_e) = \mathrm{sign}(v_{rel}), \quad x_{ss}(w_{rel}) \leq |x_e|, \end{cases} \tag{3}$$

where for $x_{ba} < |x_e| < x_{ss}(w_{rel})$:

$$\widehat{\alpha}(x_e, v_{rel}) = \frac{1}{2}\left(1 + \sin\left(\pi \frac{|x_e| - \left(\frac{x_{ss}(v_{rel}) + x_{ba}}{2}\right)}{x_{ss}(v_{rel}) - x_{ba}}\right)\right). \tag{4}$$



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PHS Formulation

Choosing the dissipation variables

$$\begin{aligned} w_{\mathrm{rel}} &= \quad v_{\mathrm{rel}} &&= \dot{x}_{\mathrm{e}} + v_{\mathrm{p}}, \\ w_{\mathrm{p}} &= \quad k_{\mathrm{e}} x_{\mathrm{e}} &&= h_{\mathrm{e}}'(x_{\mathrm{e}}), \end{aligned}$$

the Dupont model of friction reads:

$$\begin{cases} f_{i \to s} & = k_{\mathrm{e}} x_{\mathrm{e}} + (\sigma_{\mathrm{c}} + \sigma_{\mathrm{f}}) w_{\mathrm{rel}} - \sigma_{\mathrm{c}}\, r_{\mathrm{Lu}}(w_{\mathrm{rel}})\, w_{\mathrm{p}} \\ \frac{\mathrm{d} x_e}{\mathrm{d} t} & = v_{\mathrm{rel}} - r_{Du}(x_e, w_{rel}) w_{\mathrm{p}}, \end{cases} \tag{5}$$

where $r_{Du}(x_e, w_{rel}) \triangleq \alpha(x_e, w_{rel}) \frac{|w_{rel}|}{f_{ss}(w_{rel})} > 0$.

PHS formulation

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The PHS formulation of the Dupont model of friction is then

$$\left(\begin{array}{c} \frac{\mathrm{d}x_\mathrm{e}}{\mathrm{d}t} \\ \hline w_\mathrm{rel} \\ w_\mathrm{p} \\ \hline f_{\mathrm{i}\to\mathrm{s}} \\ f_{\mathrm{i}\to\mathrm{c}} \end{array}\right) = \left(\begin{array}{c|cc|cc} 0 & 0 & -1 & -1 & +1 \\ \hline 0 & 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & 0 & 0 \\ \hline +1 & +1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} h'_\mathrm{e}(x_\mathrm{e}) \\ \hline z_\mathrm{rel}(w_\mathrm{rel}, w_\mathrm{p}; x_\mathrm{e}) \\ z_\mathrm{p}(w_\mathrm{rel}, w_\mathrm{p}; x_\mathrm{e}) \\ \hline v_\mathrm{s} \\ v_\mathrm{c} \end{array}\right), \tag{6}$$

with dissipation variables $\mathrm{w} = (w_\mathrm{rel}, w_\mathrm{p})^\intercal$, dissipation function $\mathrm{z}(\mathrm{w}; x_\mathrm{e}) = R(\mathrm{w}; x_\mathrm{e})\,\mathrm{w}$ where

$$R(\mathrm{w}; x_\mathrm{e}) = \begin{pmatrix} \sigma_c + \sigma_f & -\sigma_c\, r_{Du}(x_e, w_{rel}) \\ 0 & r_{Du}(x_e, w_{rel}) \end{pmatrix} \succeq 0.$$

Remark: (i) the dissipated power is $\mathrm{w}^\intercal \mathrm{z}(\mathrm{w}; x_\mathrm{e}) \geq 0$, $\forall x_\mathrm{e}$. (ii) Generic model (Dalh for ..., Lugre for ...)

PHS of the bowed string

The finite dimensional model used for the simulations is obtained by

1. Projecting the PHS of the string on the modal basis $e_k(\xi) = \sqrt{\frac{2}{L}} \sin(\frac{\kappa\pi\xi}{L})$,
2. Connecting the ouptput of the string (velocity) to the input of the interaction model and the output of the interaction model (force) to the input of the string.

$$
\left(\begin{array}{c}
\frac{\mathrm{d}Q}{\mathrm{d}t}(t) \\
\frac{\mathrm{d}P}{\mathrm{d}t}(t) \\
\frac{\mathrm{d}x_{\mathrm{e}}}{\mathrm{d}t} \\ \hline
W_{\mathrm{s}}(t) \\
w_{\mathrm{rel}} \\
w_{\mathrm{p}} \\ \hline
f_{\mathrm{i}\to\mathrm{c}}
\end{array}\right)
=
\left(\begin{array}{ccc|ccc|c}
0 & \frac{1}{\mu}\mathbf{I}_K & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{\mu}\mathbf{I}_K & 0 & \frac{-1}{\mu}E(\xi_B) & \frac{-1}{\mu}\mathbf{I}_K & \frac{-1}{\mu}E(\xi_B) & 0 & 0 \\
0 & \frac{1}{\mu}E^T(\xi_B) & 0 & 0 & 0 & -1 & -1 \\ \hline
0 & \frac{1}{\mu}\mathbf{I}_K & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\mu}E^T(\xi_B) & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & +1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & +1 & 0 & +1 & 0 & 0
\end{array}\right)
\left(\begin{array}{c}
T_0 D_k^2 Q(t) \\
\mu P(t) \\
h'_{\mathrm{e}}(x_{\mathrm{e}}) \\ \hline
aP(t) \\
z_{\mathrm{rel}}(w_{\mathrm{rel}}, w_{\mathrm{p}}; x_{\mathrm{e}}) \\
z_{\mathrm{p}}(w_{\mathrm{rel}}, w_{\mathrm{p}}; x_{\mathrm{e}}) \\ \hline
v_{\mathrm{c}}
\end{array}\right), \quad (7)
$$

where $E^T(\xi) = (e_1(\xi)e_2(\xi)\dots e_K(\xi))$ and ξ_B denotes the position of the bow along the string.

Time discretisation

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(i) Time derivative $\rightarrow$ Euler explicit

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) \approx \frac{x(k+1)-x(k)}{T} = \frac{\delta x}{T},$$

where T is the discrete time step.

(ii) Gradient $\nabla H(x)$ $\rightarrow$ the discrete gradient[4]

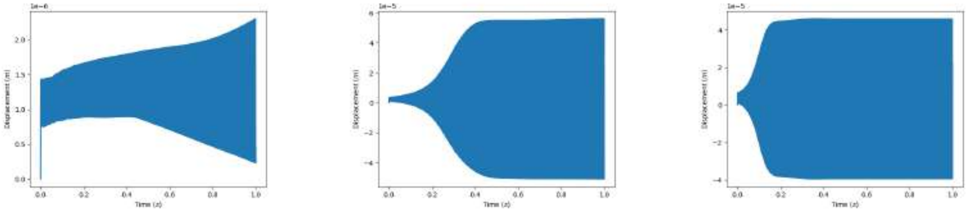
$$\nabla_d H(x, x+\delta x) = \frac{H(x+\delta x)-H(x)}{\delta x}\frac{\delta x}{T}.$$

In the case of a linear system (quadratic hamiltonian: $H = \frac{1}{2}x^T Qx$), this is:

$$\nabla_d H(x, x+\delta x) = \mathrm{Q}\left(x(k) + \frac{\delta x(k)}{2}\right).$$

[4] T. Itoh, K. Abe, Hamiltonian-conserving discrete canonical equations based on variational difference quotients, Journal of Computational Physics 76 (1)(1988) 85–102

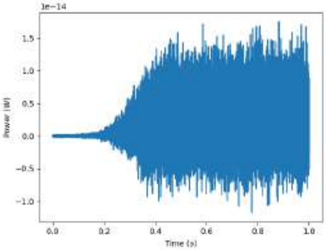
Sound synthesis results



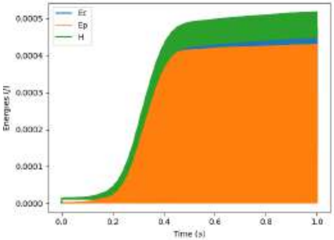
(a) $\xi_B = L/10$ (b) $\xi_B = L/4$ (c) $\xi_B = L/2$

Figure: Displacement of the string at the interaction point for differents bowing points.

Sound synthesis results



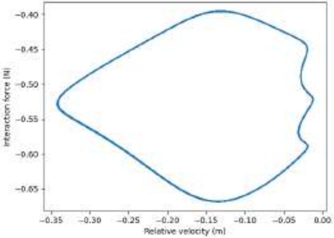
(a) Power balance ($\frac{H(x(t_{n+1})-x(t_n)}{T} + P_D - P_{ext}$)



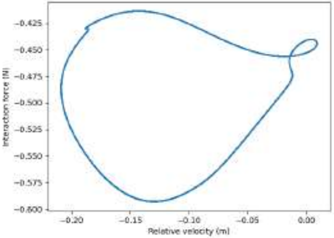
(b) Energies

Figure: Power balance is preserved as it can be seen in the first column which present the difference signal between the variation of the Hamiltonian and the power damped minus the power of the sources.

Sound synthesis results



(a) $\xi_B = L/4$



(b) $\xi_B = L/2$

Figure: Interaction force as a function of the relative velocity: the stick-slip motion is visible.

Conclusion

Results

- Generic interaction model (collisions, friction) in the PHS formalism
- Coupling resolution in the PHS model
- Passive-guaranteed numerical simulation
- Self-oscillations emergence depends on several physical and interaction parameters (bowing point, velocity and force of the bow)

Perspectives

- Application to nonlinear resonators (e.g. string or plates in large deformations)
- Optimisation of numerical schemes and code with an aim for real-time synthesis